Chapter 8 Applications of Straight-Line Graphs ⊕
Reteach Worksheet
8.1 Reducing Equations to Linear Form

Name: ______________________________ Date: _______________

Class: _______________

Notes

1 To transform a non-linear equation in x and y to the linear form $Y = mX + c$, follow these steps:
- **(a)** Perform operations on both sides of the equation to obtain the constant term c.
- **(b)** Compare the equation with the linear form to find X and Y as a function of x and/or y.

Example 1

Express the following equations in the form $Y = mX + c$, where X and Y are each functions of x and/or y, and m and c are constants. State the value of m and of c in each case.

(a) $y = 5x\sqrt{x} + 10$ **(b)** $8y - x = 7xy$

(c) $y = -16x^2 + \frac{100}{x}$ **(d)** $y^2 = \frac{1000}{x^5}$

Solution

(a) *Substitute the terms in y and x with Y and X respectively.*

$y = 5x\sqrt{x} + 10$

$Y = 5X + 10$, *where* $Y = 5$ and $X = x\sqrt{x}$.

Hence $m = 5$ and $c = 10$.

(b) *Apply division to eliminate unknowns and obtain a constant term.*

$$8y - x = 7xy$$
$$\frac{8y}{xy} - \frac{x}{xy} = \frac{7xy}{xy}$$
$$\frac{8}{x} - \frac{1}{y} = 7$$
$$\frac{1}{y} = \frac{8}{x} - 7$$

Remark

The linear form is not unique. You can also divide $8y - x = 7xy$ by either x or y to obtain a constant term.

Substitute the terms in y and x with Y and X respectively.

$Y = 8X - 7$, where $Y = \frac{1}{y}$ and $X = \frac{8}{x}$.

Hence $m = 8$ and $c = -7$.

(c) *Eliminate the denominator by multiplying the term throughout.*

$$y = -16x^2 + \frac{100}{x}$$
$$y \cdot x = -16x^2 \cdot x + \frac{100}{x} \cdot x$$
$$xy = -16x^3 + 100$$

Substitute the terms in y and in x with Y and X respectively.

$Y = 16X + 100$, where $Y = xy$ and $X = -x^3$.

Hence $m = -16$ and $c = 100$.

(d) *Take common logarithms on both sides. Simplify to obtain a constant term.*

$$y^2 = \frac{1000}{x^5}$$
$$\lg y^2 = \lg\left(\frac{1000}{x^5}\right)$$
$$2\lg y = \lg 1000 - \lg x^5$$
$$2\lg y = \lg 10^3 - 5\lg x$$
$$2\lg y = -5\lg x + 3\lg 10$$
$$\lg y = -\frac{5}{2}\lg x + \frac{3}{2}$$

Substitute the terms in y and x with Y and X respectively.

$$Y = -\frac{5}{2}X + \frac{3}{2}$$

$Y = -\frac{5}{2}x + \frac{3}{2}$, where $Y = \lg y$ and $X = \lg x$.

Hence $m = -\frac{5}{2}$ and $c = \frac{3}{2}$.

Solve.

1A Express the following equations in the form $Y = mX + c$, where X and Y are each functions of x and/or y, and m and c are constants. State the value of m and of c in each case.

(a) $\frac{y}{x} = \frac{1}{8}x^3 - 2$

(b) $6y = 2xy - 3x$

(c) $y = -2x + \frac{9}{\sqrt{x}}$

(d) $y = 90e^{-\frac{9x}{100}}$

Solution

(a) *Substitute the terms in y and x with Y and X respectively.*

$$\frac{y}{x} = \frac{1}{8}x^3 - \frac{1}{2}$$
$$Y = \frac{1}{8}X - \frac{1}{2}$$

Hence $m = \frac{1}{8}$, $c = -\frac{1}{2}$.

(b) *Apply division to eliminate unknowns and obtain a constant term.*

$$y = -5x^2\sqrt{x} + 12\sqrt{x}$$
$$\frac{y}{\sqrt{x}} = \frac{-5x^2\sqrt{x}}{\sqrt{x}} + \frac{12\sqrt{x}}{\sqrt{x}}$$
$$\frac{y}{\sqrt{x}} = -5x^2 + 12$$

Substitute the terms in y and x with Y and X respectively.

$Y = -5X + 12$, where $Y = \frac{y}{\sqrt{x}}$ and $X = x^2$

Hence $m = -5$ and $c = 12$.

(c) *Eliminate the denominator by multiplying the term throughout.*

$$y = -2x + \frac{9}{\sqrt{x}}$$
$$y \cdot \sqrt{x} = -2x \cdot \sqrt{x} + \frac{9}{\sqrt{x}} \cdot \sqrt{x}$$
$$y\sqrt{x} = -2x\sqrt{x} + 9$$

Substitute the terms in y and x with Y and X respectively.

$Y = -2X + 9$

Hence $m = -2$ and $c = 9$.

(d) *Take natural logarithms on both sides. Simplify to obtain a constant term.*

$$y = 90\mathrm{e}^{-\frac{9x}{100}}$$
$$\ln y = \ln\left(90\mathrm{e}^{-\frac{9x}{100}}\right)$$
$$\ln y = \ln 90 + \ln \mathrm{e}^{-\frac{9x}{100}}$$
$$\ln y = \ln 90 - \frac{9}{100}x$$
$$\ln y = -\frac{9}{100}x + \ln 90$$

Substitute the terms in y and x with Y and X respectively.

$Y = -\frac{9}{100}X + \ln 90$, where $Y = \ln y$ and $X = x$.

Hence $m = -\frac{9}{100}$ and $c = \ln 90$.

1B Express the following equations in the form $Y = mX + c$, where X and Y are each functions of x and/or y, and m and c are constants. State the value of m and of c in each case.

(a) $\lg y = \frac{2x}{y} - \frac{3}{2}$

(b) $3y^2 = x - 9y$

(c) $y = \frac{3x^2 + 5}{x}$

(d) $y = \frac{1}{10}(x+1)^3$

Solution

(a) $\lg y = \frac{2x}{y} - \frac{3}{2}$

$Y = 2X - \frac{3}{2}$, where $Y = \lg y$ and $X = \frac{x}{y}$.

(b) $3y^2 = x - 9y$

$\frac{3y^2}{3y} = \frac{x}{3y} - \frac{9y}{3y}$

$y = \frac{1}{3} \cdot \left(\frac{x}{y}\right) - 3$

$Y = \frac{1}{3}X - 3$, where $Y = y$ and $X = \frac{x}{y}$.

Hence $m = \frac{1}{3}$ and $c = -3$.

(c) $$y = \frac{3x^2 + 5}{x}$$

$$y \cdot x = \frac{3x^2 + 5}{x} \cdot x$$

$$xy = 3x^2 + 5$$

$Y = 3X + 5$, where $Y = xy$ and $X = x^2$.

Hence $m = 3$ and $c = 5$.

(d) $$y = \frac{1}{10}(x+1)^3$$

$$\lg y = \lg\left[\frac{1}{10}(x+1)^3\right]$$

$$\lg y = \lg\frac{1}{10} + \lg(x+1)^3$$

$$\lg y = \lg\frac{1}{10} + 3\lg(x+1)$$

$$\lg y = 3\lg(x+1) + \lg(10^{-1})$$

$$\lg y = 3\lg(x+1) - 1$$

$Y = 3X - 1$, where $Y = \lg y$ and $X = \lg(x+1)$.

Hence $m = 3$ and $c = -1$.

Example 2

The variables x and y are related by the equation $2y^2 = x - 6y$.

(i) When y is plotted against $\frac{x}{y}$, a straight line is obtained. Find the equation of this line in the form $Y = mX + c$, where m and c are constants.

(ii) Draw the graph of the line in (i), indicating clearly the X- and Y-intercepts.

Solution

(i) *Identify the terms of x and/or y to be represented by x and y respectively.*

When y is plotted against $\frac{x}{y}$, a straight line of the form $Y = mX + c$ is obtained.

$\Rightarrow Y = y$ and $X = \frac{x}{y}$

Express the original equation relating x and y in the form Y = mX + c.

Divide the equation $2y^2 = x - 6y$ by $2y$ to obtain Y (i.e. y).

$$\frac{2y^2}{2y} = \frac{x}{2y} - \frac{6y}{2y}$$

$$y = \frac{x}{2y} - 3$$

Hence the equation is $Y = \frac{1}{2}X - 3$, where $Y = y$ and $X = \frac{x}{y}$.

From the equation, the gradient $m = \frac{1}{2}$ and the y-intercept $c = -3$.

(ii) *Find the X-intercept.*

Substitute $Y = 0$ into the equation to obtain the X-intercept.

$0 = \frac{1}{2}X - 3$

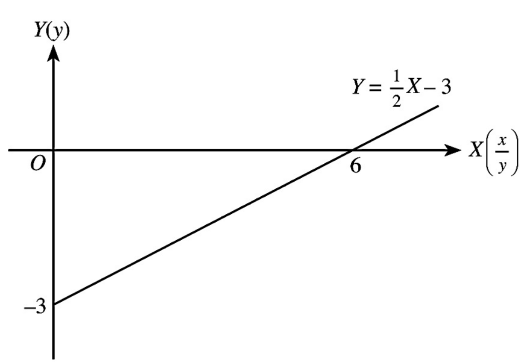
$3 = \frac{1}{2}X$

$X = 6$

Remark

From (i), we know that the straight line passes through (0, –3). In order to draw a straight line, we need another point.

Draw the graph.



Solve.

2A The variables x and y are related by the equation $y = \dfrac{x+5}{x+2}$, $x \neq -2$.

(i) When y is plotted against $x - xy$, a straight line is obtained. Find the equation of this line in the form $Y = mX + c$, where m and c are constants.

(ii) Draw the graph of the line in (i), indicating clearly the X- and Y-intercepts.

Solution

(i) *Identify the terms of x and/or y to be represented by x and y respectively.*

When y is plotted against $x - xy$, a straight line of the form $Y = mX + c$ is obtained.
$\Rightarrow Y = y$ and $X = x - xy$

Express the original equation relating x and y in the form Y = mX + c.

Multiply both sides of the equation by $x + 2$.

$$y \cdot (x+2) = \frac{x+5}{x+2} \cdot (x+2)$$
$$xy + 2y = x + 5$$
$$2y = x - xy + 5$$
$$y = \frac{1}{2}(x - xy) + \frac{5}{2}$$

Hence the equation is $Y = \dfrac{1}{2}X + \dfrac{5}{2}$, where $Y = y$ and $X = x - xy$.

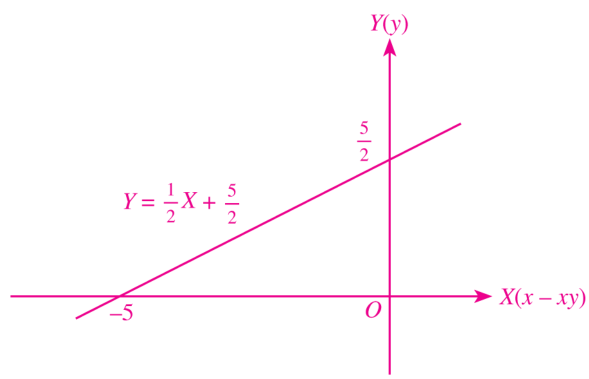
From the equation, the gradient $m = \dfrac{1}{2}$ and the y-intercept $c = \dfrac{5}{2}$.

(ii) *Find the X-intercept.*

Substitute $Y = 0$ into the equation to obtain the X-intercept.

$$Y = \frac{1}{2}X + \frac{5}{2}$$
$$0 = \frac{1}{2}X + \frac{5}{2}$$
$$\frac{1}{2}X = -\frac{5}{2}$$
$$X = -5$$

Draw the graph.



2B The variables x and y are related by the equation $y = \dfrac{3x-4}{x+3}$, where $x \neq -3$.

(i) When $y - x$ is plotted against xy, a straight line is obtained. Find the equation of this line in the form $Y = mX + c$, where m and c are constants.

(ii) Draw the graph of the line in (i), indicating clearly the X- and Y-intercepts.

Solution

(i) When $y - x$ is plotted against xy, a straight line of the form $Y = mX + c$ is obtained.

$\Rightarrow Y = y - x$ and $X = xy$

Divide the equation $y = \dfrac{3x-4}{x+3}$ by $x + 3$.

$$y = \frac{3x-4}{x+3}$$
$$xy + 3y = 3x - 4$$
$$3y - 3x = -xy - 4$$
$$3(y - x) = -xy - 4$$
$$y - x = -\frac{1}{3}xy - \frac{4}{3}$$

Hence the equation is $Y = -\dfrac{1}{3}X - \dfrac{4}{3}$, where $Y = y - x$ and $X = xy$.

From the equation, the gradient $m = -\dfrac{1}{3}$ and the y-intercept $c = -\dfrac{4}{3}$.

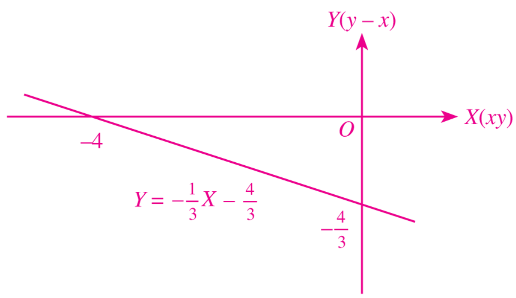
(ii) Substitute $Y = 0$ into the equation to obtain the X-intercept.

$$Y = -\frac{1}{3}X - \frac{4}{3}$$

$$0 = -\frac{1}{3}X - \frac{4}{3}$$

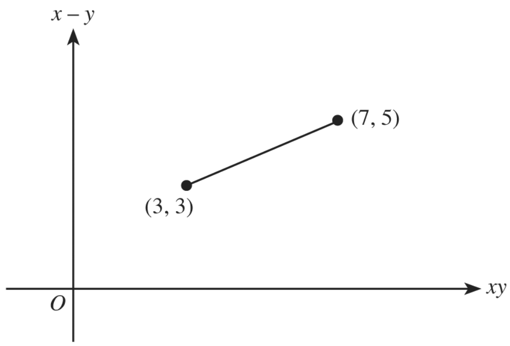
$$\frac{1}{3}X = -\frac{4}{3}$$

$$X = -4$$



Example 3

The diagram shows the straight-line graph of $x - y$ against xy, passing through the points (3, 3) and (7, 5).



The variables x and y are connected by the equation $y = \frac{ax+b}{x+2}$, where a and b are constants. Find the value of a and of b.

Solution

Identify the terms of x and/or y to be represented by x and y respectively.

When $x - y$ is plotted against xy, the line is of the form $Y = mX + c$, where $Y = x - y$ and $X = xy$.

Find the gradient of the line.

The line passes through (3, 3) and (7, 5).
So the gradient, m, is $\frac{5-3}{7-3} = \frac{1}{2}$.

Substitute the value of m and the coordinates of one of the given points into y = mx + c to obtain the value of c.

Substitute $X = 3$, $Y = 3$ and $m = \frac{1}{2}$ into $Y = mX + c$.

$3 = \frac{1}{2}(3) + c$

$c = \frac{3}{2}$

Write the equation in the form Y = mX + c.

So $Y = \frac{1}{2}X + \frac{3}{2}$ or $x - y = \frac{1}{2}xy + \frac{3}{2}$.

Rewrite the equation in the form $y = \frac{ax+b}{x+2}$.

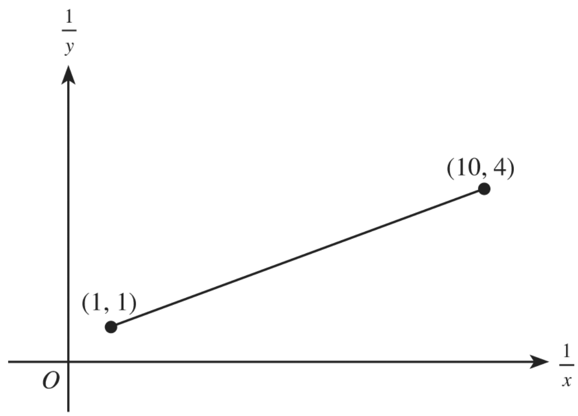
$$\begin{aligned} x - y &= \frac{1}{2}xy + \frac{3}{2} \\ x - \frac{3}{2} &= \frac{1}{2}xy + y \\ x - \frac{3}{2} &= y\left(\frac{1}{2}x + 1\right) \\ y &= \frac{x - \frac{3}{2}}{\frac{1}{2}x + 1} \\ y &= \frac{2x-3}{x+2} \end{aligned}$$

Compare to the given equation to obtain value of a and of b.

Hence $a = 2$ and $b = -3$.

Solve.

3A The diagram shows the straight-line graph of $\frac{1}{y}$ against $\frac{1}{x}$, passing through the points (1, 1) and (10, 4).



The variables x and y are connected by the equation $ax + by = xy$, where a and b are constants. Find the value of a and of b.

Solution

Identify the equivalent functions of X and Y respectively, in terms of x and/or y.

When $\frac{1}{y}$ is plotted against $\frac{1}{x}$, the line is of the form $Y = mX + c$, where $Y = \frac{1}{y}$ and $X = \frac{1}{x}$.

Find the gradient of the line.

The line passes through (1, 1) and (10, 4).

So the gradient, m, is $\frac{4-1}{10-1} = \frac{3}{9} = \frac{1}{3}$.

Substitute the value of m and the coordinates of one of the given points into y = mx + c to obtain the value of c.

Substitute $X = 1$, $Y = 1$ and $m = \frac{1}{3}$ into $Y = mX + c$.

$1 = \frac{1}{3}(1) + c$

$c = \frac{2}{3}$

Write the equation in the form Y = mX + c.

So $Y = \frac{1}{3}X + \frac{2}{3}$ or $\frac{1}{y} = \frac{1}{3} \cdot \frac{1}{x} + \frac{2}{3}$

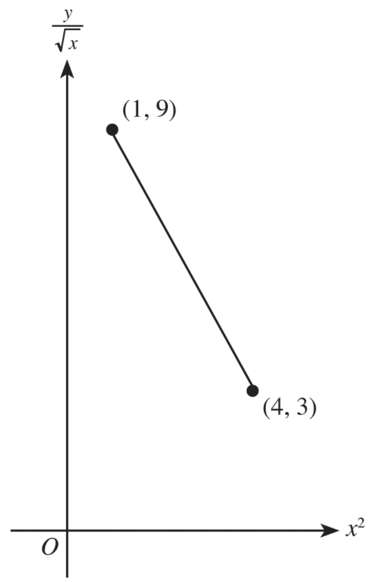
Rewrite the equation in the form ax + by = xy.

$$\frac{1}{y} = \frac{1+2x}{3x}$$
$$3x = y + 2xy$$
$$\frac{3}{2}x = \frac{1}{2}y + xy$$
$$\frac{3}{2}x - \frac{1}{2}y = xy$$

Compare to the given equation to obtain value of a and of b.

Hence $a = \frac{3}{2}$ and $b = -\frac{1}{2}$.

3B The diagram shows the straight-line graph of $\frac{y}{\sqrt{x}}$ against x^2, passing through the points (1, 9) and (4, 3).



The variables x and y are connected by the equation $y = (ax^2 + b)\sqrt{x}$, where a and b are constants. Find the value of a and of b.

Solution

When $\frac{y}{\sqrt{x}}$ is plotted against x^2, the line is of the form $Y = mX + c$, where $Y = \frac{y}{\sqrt{x}}$ and $X = x^2$.

The line passes through (1, 9) and (4, 3).

So the gradient, m, is $\frac{3-9}{4-1} = -2$.

Substituting $X = 1$, $Y = 9$ and $m = -2$ into $Y = mX + c$,

$9 = -2(1) + c$

$c = 11$

So $Y = -2X + 11$ or $\frac{y}{\sqrt{x}} = -2x^2 + 11$.

$$\frac{y}{\sqrt{x}} \cdot \sqrt{x} = -2x^2 \cdot \sqrt{x} + 11 \cdot \sqrt{x}$$

$$y = \left(-2x^2 + 11\right)\sqrt{x}$$

Hence $a = -2$ and $b = 11$.